We are interested in the problem of finding the smallest enclosing sphere for a set of points in $\mathbb{R}^n$. Megiddo showed how to use linear programming to provide a linear time algorithm to this problem in $\mathbb{R}^2$ and $\mathbb{R}^3$ in 1983.

This note contains some supplementary material to accompany his paper "Linear-Time Algorithms for Linear Programming in $\mathbb{R}^3$ and Related Problems."

1 Basic definitions

Let $p, q \in \mathbb{R}^n$ with the components of p denoted $p = [p_1, \ldots p_n]^T$. Then $(p, q) = \sum_{i=1}^{n} p_i q_i$ is the **inner product** of p and q, and the **Euclidean norm** of p is $\|p\| = \sqrt{(p,p)} = \sqrt{\sum_{i=0}^{n} p_i^2}$. The **Euclidean distance** between p and q is defined as $\|p - q\|$.

2 Convexity

A set $A \subseteq \mathbb{R}^n$ is **convex** if for all $s, t \in A$ and $\lambda \in [0,1]$, the point $\lambda s + (1-\lambda)t$ also lies in A.

A real-valued function f on an open interval $A = (a, b) \subseteq \mathbb{R}$ is **convex** if for $\lambda \in [0,1]$, and $s, t \in (a, b)$ the following inequality holds:

$$f((1-\lambda)s + \lambda t) \leq (1-\lambda)f(s) + \lambda f(t) \tag{1}$$

Graphically, this requires that for any $x \in (s,t)$, $(x, f(x))$ lies below the line connecting $(t, f(t))$ and $(s, f(s))$. Taking the slope function $\phi(x, y) = (f(x) - f(y))/(x - y)$ (where $x \neq y$), the above definition is equivalent to the requirement that for all $x_0 \in (a, b)$, $\phi(x_0, y)$ is increasing as a function of y.

A function $h : \mathbb{R}^n \Rightarrow \mathbb{R}$ is also **convex** if it satisfies (1) (now taking A to be a convex open subset of $\mathbb{R}^n$). This is equivalent to the requirement that h restricted to any line segment in $\mathbb{R}^n$ is a convex function. [Note that the direction of the parameterization does not matter.]

Supposing that f and g are convex over a convex open set A, we have the following:

1. f and g are continuous on A. [See Rudin Chapter 3 or Artin Chapter 1]

2. $f + g$ is convex on A. [Add inequality 1 for f and g]

3. $\max(f, g)$ is convex on A. [Fix any triple s, t, λ and examine the cases]

3 Defining a sphere with a minimal number of points

Lemma 1 (Separating Hyperplanes) *Let $u, v \in \mathbb{R}^n$. Then the set of all points equally distant from u and v is a (hyper-)plane perpendicular to and bisecting $\overline{uv}$.*

Corollary 1 *If two points u and v lie upon a circle with center c, then c lies upon the perpendicular bisector of $\overline{u\,v}$.*

Corollary 2 *If a circle is specified by n points upon its edge, any three (say a, b, c) will fully specify the circle: Its center is the intersection of perpendicular bisectors of $\overline{a\,b}$ and $\overline{b\,c}$.*

Generalizations

- For any given set G of n linearly independent points, there are infinitely many n spheres with said surface points. [G for **general position**]
- $n+1$ linearly independent surface points uniquely determine an n sphere.

Linear independence is vital here. Examples:

1. $D = \{(0,-1),(0,0),(0,1)\}$. There is no circle that passes through all the points of D (**degenerate**), as Lemma 1 implies that a point c equally distant from s_1 and s_2 must be of the form $(c_x, -0.5)$. c equally distant from s_2 and s_3 implies $c = (c_x, 0.5)$.
2. $D = \{(0,1,0),(1,0,0),(0,-1,0),(-1,0,0)\}$. Although these four points do determine a circle with center at the origin, they determine infinitely many 3-spheres: take the centers to be $(0,0,z)\,\forall z \in \mathbb{R}$.

4 Given an enclosing sphere, finding a smaller sphere

Lemma 2 (Characterization of Minimal Enclosing Circles) *Let $P \subset \mathbb{R}^2$ be a non-empty, finite set of points in the plane. Then a minimal circle enclosing P is specified either by: (1) the two furthest separated points u, v which lying upon the circle's edge form a diameter of the circle, or (2) three points of P which lie upon the circle's edge.*

Taking the two maximally separated points $u, v \in P$, it is clear that a minimal enclosing circle cannot have a diameter less than $\|u - v\|$. Thus if the circle C centered between u and v with diameter $\|u - v\|$ does enclose all points of P, then it is a minimal enclosing circle.

It is not necessarily true that two maximally separated points will both lie upon any minimal circle bounding a set of points. Take as an example

$$P = \{(-1,0),(1,0),(7/8,1/2),(7/8,-1/2),(-0.313,0.96),(-0.313,-0.96)\}$$

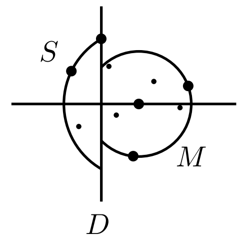
The first two points of P form the most separated pair of P. Surprisingly, though, the minimal enclosing circle touches *neither of them*. Instead, it is determined by these three points of P upon its edge: $\{(7/8,-1/2),(-0.313,0.96),(-0.313,-0.96)\}$.

Now let us complete our proof of the lemma by showing the following sublemma:

Lemma 3 (Circle shrinking lemma) *Let C be a circle with center c and radius $r > 0$ enclosing a finite set of points P. Let $E \subseteq P$ be the subset lying upon C's edge. If a line L through the c can be found such that E lies entirely to one side of L (or equivalently, c does not lie in the convex hull of E), then a smaller circle can be found enclosing P.*

We will construct two circular segments centered at c meeting at a dividing line D which together contain P. We will then construct a circle smaller than C containing these segments.

Let $S = \{s_0, \ldots\} \subseteq P$ be the subset of P maximally distant from c, all at distance r_S. Let $L^\perp$ be the line perpendicular to L passing through c and let D be the line parallel to L intersecting the point of S closest to c. Since S is finite, D intersects $L^\perp$ at a point d some distance $\epsilon > 0$ from c.



Take $T \subseteq P$ be the set of all points in the half plane determined by D containing c (excluding D), and let $M \subseteq T$ be the subset of T maximally distant from c, all at distance $r_M < r_S \leq r$.

Take the two circular segments centered at c, DS and DM. Note that for any point x on $\overline{cd}$, the furthest points in DS from x are its corners, and the furthest point in DM from x is the intersection of the arc M with $L^\perp$. Hence, any point x within $r - r_M$ of c on $\overline{cd}$ yields a circle with radius $r' < r$ containing all points of P.

With Lemma (3) proved, Lemma (2) follows since a circle is a candidate for minimal enclosing circle only if its center lies within the convex hull of the subset of P upon its edge. In two dimensions, we have shown that three linearly independent points are sufficient to fully define a circle.

Generalization

Let $P \subset \mathbb{R}^n$ be a non-empty, finite set of points. Then the center of any smallest sphere enclosing P must lie within the convex hull of the points of P upon its edge.

Given the results heretofore, the minimal enclosing sphere of n points in d dimensions can be found in $O(n^{d+2})$ time by examining the spheres defined by all combination of $d+1$ or fewer of the n points.

Lemma 4 (Uniqueness) *The smallest sphere enclosing a set of points P is unique.*

Let there be two spheres of radius r which enclose a set of points P centered at points c_0 and c_1 which are $\delta \geq 0$ apart. Then the points P are also enclosed by the sphere of radius $\sqrt{r^2 - \delta^2}$ about the point $\frac{c_0+c_1}{2}$.

5 Notes on Megiddo's paper

"The extreme point problem in $\mathbb{R}^2$ is reducible to linear programming in $\mathbb{R}^1$, modeled as the problem of finding a straight line through p_0 that has all the points $p_1 \ldots p_n$ on one side of it." p. 759

Megiddo shows how this is done in the appendix: First apply an affine transform mapping $p_0 \to (0,0)$ and $p_1 \to (1,0)$. For each point $p_2 \ldots p_n = (a_2, b_2) \ldots (a_n, b_n)$:

If $a_i > 0$, add the constraint $\alpha < b_i/a_i$.

If $a_i < 0$, add the constraint $\alpha > b_i/a_i$.

If $a_i = 0$ and $b_i < 0$ return FALSE (p_0 is not an extreme point).

"Analogously, if $S_g < 0$ than $x^* > x$." p. 761 (typo).

"Moreover, if the maximum is attained at more than one index then x is a local minimum" p. 763

Suppose the maximum is attained at indices i and j. then if we move $x \to x'$ away from i, then $r(x') >= r_i(x') > r_i(x)$. if we move $x \to x'$ away from j then $r(x') >= r_j(x') > r_j(x)$.

"Obviously ... Obviously ..." p. 767

These claims are much more obvious in light of the circle shrinking lemma.

"It is essential to note that $f(x,y) = \max\{(x-a_i)^2 + (y-b_i)^2 : 1 \le i \le n\}$ is convex, not only in each variable, but also as a function of two variables. This also implies that the function $h(y) = \min_x f(x,y)$ is also convex" p. 767

The reasoning here takes a little work for me, since the definitions I can find regarding convex functions of two variables characterize them by their behavior on line segments.

Take $y_1, y_3 \in \mathbb{R}$ and $\lambda \in (0,1)$. Take $y_2 = (1-\lambda)y_1 + \lambda y_3$, and let x_i be the value of x minimizing $h(x, y_i)$.

Let $x_2' = (1-\lambda)x_1 + \lambda x_3$. Then we have:

$$h(y_2) \le f(x_2', y_2) \le (1-\lambda)h(y_1) + \lambda h(y_3)$$

The first inequality stems from the definition of h, and the second from the convexity of f.

"Consider the linear transformation that takes the x-axis to the line $y = \alpha_m x \ldots$ and leaves the y-axis fixed." 4.3 p. 768

This shear is not what Megiddo wants (it does not preserve angles or distances). What he really means is a rotation of the plane taking the line $y = \alpha_m x$ to the line $y = 0$.

"It follows (see Fig. 5) that one of the points defining L_i, namely, the one that lies "southwest" of it, can be dropped since the other point will be at least as far from the center." p. 769

This reasoning works only with strict inequality: you have to replace "at least as far" with "farther."